3-Dimensional Wave Equation Based Analysis of Electromagnetic Wave

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Abstract: In this paper, I will examine the solution to the wave equation in 3 dimensions and justify the method of separation of variables through which I arrived at the solution. A 3-dimensional wave equation is of applied significance in describing wavelike phenomena; common examples can include atmospheric waves, electromagnetic waves, and gravitational waves

1. Introduction

The 3-dimensional (3D) wave equation transpires to be a linear, homogeneous partial differential equation (PDE) [1,2]. By considering arrays of sequenced oscillators, or otherwise, one obtains the general structure of the 3 -dimensional wave equation. Solutions of wave equations are of evident significance, for they provide a theoretical insight on understanding the behaviours of travelling waves in specific mediums [3]. Some of the wave equations can be solved analytically by separation of variables [4]. Other approaches to obtain the analytical solution include the method of integral transform [2] (i.e., the Laplace transform and/or the Fourier transform), both of which reflect a genuine sense of ingeniousness in reductionism and analytic philosophy. Routines of solving the wave equations numerically in the finite-difference approach are also widely investigated, where the degree of accuracy and analytical predictions of grid dispersion effect are examined [5] .

The research on the analytical and numerical solutions to the wave equations has been applied interdisciplinarily, into fields of engineering, physics and geology science, to name a few. Through examining the wave equation, seismic illumination analysis provides a novel and flexible approach in tackling complex wave models with nontrivial acquisition and target geometry [6] Innovative and convenient methods are developed in electronics, optics and acoustics [7,8,9]. Not confined to its engineering values, the solutions of the wave equations can contribute as a pedagogical tool. Topics like light propagation and basic quantum mechanics can be explained by solving the wave equations as a solid mathematical foundation[10]. The strategy adopted in this paper involves less complicated technique which makes it more conceptually comprehensible and academically accessible by many, but yields at the same time fruitful and valid results which may be used for specific purposes. Such elegant simplicity appreciated by many is, logically speaking, an inevitable concomitant of the underlying principles and fundamental guidelines of separation of variables - divide and rule.

In this letter, we solve the 3D linear wave equation in a Cartesian coordinate system and apply the analysis to an electromagnetic wave propagation problem. The key aim of this paper is to reflect properties through rigorous derivation alone, from which we further extrapolate known identities to inspect the validity of the presented solution.

2. Theory

The 3-dimensional wave equation $U(x, y, z, t)$ can be generalised as follows

$$\frac{\partial^2 U}{\partial t^2} = \alpha^2 \nabla^2 U,\ (x, y, z) \in \mathbb{R}^3, t > 0, \tag{1}$$

where the scalar quantity α indicates the speed with which the wave is propagating and ∇^2 represents the Laplace operator

$$\nabla^2 U = \frac{\partial^2 U}{\partial^2 x} + \frac{\partial^2 U}{\partial^2 y} + \frac{\partial^2 U}{\partial^2 z}. \tag{2}$$

2.1 Separation of Variables in Cartesian Coordinates

Now, in order to perform separation of variables on equation (1), postulate the function $U(x, y, z, t)$ can be expressed in the form of a product of several single-variable function. To be more specific, postulate

$$U(x, y, z, t) = X(x)Y(y)Z(z)T(t). \tag{3}$$

By performing substitution back into equation (1), it follows immediately that

$$\frac{1}{\alpha^2} XYZT'' = X''YZT + XY''ZT + XYZ''T, \tag{4}$$

where the double primes indicate the ordinary second-order derivative of the single-variable functions. The primary advantage that arises subsequently is that the equation (1), which involves partial derivatives, has been suppressed deliberately into one which only consists of multiple single variable derivatives, drastically reducing the complexity and hence enabling one to adopt strategies frequently deployed in tackling ordinary differential equations. Notice that by the assumption of 3 dimensions

$$X(x)Y(y)Z(z)T(t) \neq 0. \tag{5}$$

Therefore, diving both sides of equation (4) by $X(x)Y(y)Z(z)T(t)$ gives

$$\frac{1}{\alpha^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}. \tag{6}$$

Examining functions on both sides of equation (6), one notices that the left-hand side is dependent on the variable t and the t only, whereas the right-hand side does not. Instead, the right-hand side is related to the variables x, y, and z. Nevertheless, equation (6) still holds. Hence, and for the sake of clarity and convenience, one concludes that

$$\frac{1}{\alpha^2}\frac{T''}{T} = -\lambda_4^2, \frac{X''}{X} = -\lambda_1^2, \frac{Y''}{Y} = -\lambda_2^2, \frac{Z''}{Z} = -\lambda_3^2, \tag{7}$$

where $\lambda_i,\ i = 1,2,3,4$, are the separation constants. Equation (7) have the general solutions

$$\begin{aligned} X(x) &= A_1 e^{i\lambda_1 x} + B_1 e^{-i\lambda_1 x}, \\ Y(y) &= A_2 e^{i\lambda_2 y} + B_2 e^{-i\lambda_2 y}, \\ Z(z) &= A_3 e^{i\lambda_3 z} + B_3 e^{-i\lambda_3 z}, \\ T(t) &= A_4 e^{i\alpha\lambda_4 t} + B_4 e^{-i\alpha\lambda_4 t}, \end{aligned} \tag{8}$$

where $A_i, B_i,\ i = 1,2,3,4$, are the constants determined by the boundary conditions and initial conditions and the separation constants are restricted by the relation:

$$\lambda_4^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2. \tag{9}$$

Consider the Dirichlet boundary conditions

$$X(0) = 0 = X(L_x), Y(0) = 0 = Y(L_y), Z(0) = 0 = Z(L_z), \tag{10}$$

the spatial separation constants $\lambda_1, \lambda_2, \lambda_3$ can be computed by substituting equation (8) into equation (10), and with some simple arithmetic, one gets

$$\lambda_1 = \frac{n_x \pi}{L_x}, \lambda_2 = \frac{n_y \pi}{L_y}, \lambda_3 = \frac{n_z \pi}{L_z},\ n_{x,y,z} = 1,2,3,\cdots, \tag{11}$$

The solution to the original wave equation is the superposition of the separated solutions, and the constants of the spatial part can be absorbed into the time-dependent part,

$$U(x,y,z,t) = \sum_{n_x=1,n_y=1,n_z=1}^{+\infty} \sin\left(\frac{n_x \pi x}{L_x}\right) \sin\left(\frac{n_y \pi y}{L_y}\right) \sin\left(\frac{n_z \pi z}{L_z}\right) \left(T_{\vec{n}} \cos(\phi_{\vec{n}} t) + R_{\vec{n}} \sin(\phi_{\vec{n}} t)\right), \quad (12)$$

where

$$\vec{n} = (n_x, n_y, n_z), \phi_{\vec{n}}^2 = \alpha^2 \left[\left(\frac{n_x \pi}{L_x}\right)^2 + \left(\frac{n_y \pi}{L_y}\right)^2 + \left(\frac{n_z \pi}{L_z}\right)^2 \right]. \quad (13)$$

Now, introduce initial conditions

$$U(x,y,z,0), \left.\frac{\partial U}{\partial t}\right|_{t=0}, \quad (14)$$

to illustrate the starting shape and speed of the wave, respectively. Immediately, one could get with some tedious calculations the exact values of $T_{\vec{n}}$ and $R_{\vec{n}}$ through

$$T_{\vec{n}} = \frac{8}{L_x L_y L_z} \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} U(x,y,z,0) \sin\left(\frac{n_x \pi x}{L_x}\right) \sin\left(\frac{n_y \pi y}{L_y}\right) \sin\left(\frac{n_z \pi z}{L_z}\right) dxdydz, \quad (15)$$

$$R_{\vec{n}} = \frac{8}{L_x L_y L_z \phi_{\vec{n}}} \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \left.\frac{\partial U}{\partial t}\right|_{t=0} \sin\left(\frac{n_x \pi x}{L_x}\right) \sin\left(\frac{n_y \pi y}{L_y}\right) \sin\left(\frac{n_z \pi z}{L_z}\right) dxdydz. \quad (16)$$

3. Solving Electromagnetic Waves in Linear Dielectric Media

Consider a Cartesian coordinate system, the dielectric constant $\varepsilon(x,y,z)$ is distributed as:

$$\varepsilon(x,y,z) = \begin{cases} \varepsilon_1 & x > 0, \\ \varepsilon_0 & x \leq 0 \end{cases} \quad (17)$$

The permeability is approximately homogeneous over the space,

$$\mu(x,y,z) = \mu_0. \quad (18)$$

The free space Maxwell's equations are

$$\begin{aligned} \nabla \cdot \mathrm{E}(x,y,z,t) &= 0, \\ \nabla \cdot \mathrm{B}(x,y,z,t) &= 0, \\ \nabla \times \mathrm{E}(x,y,z,t) &= -\frac{\partial}{\partial t}\mathrm{B}(x,y,z,t), \\ \nabla \cdot \mathrm{B}(x,y,z,t) &= \mu_0 \varepsilon(x,y,z) \frac{\partial}{\partial t}\mathrm{E}(x,y,z,t), \end{aligned} \quad (19)$$

where $\nabla = \frac{\partial}{\partial x}\mathrm{e}_x + \frac{\partial}{\partial y}\mathrm{e}_y + \frac{\partial}{\partial z}\mathrm{e}_z$ is the differential operator and vector functions $\mathrm{E}(x,y,z,t)$ and $\mathrm{B}(x,y,z,t)$ are the electric and magnetic field. The free space Maxwell's equations give the electromagnetic wave equations

$$\begin{aligned} \nabla^2 \mathrm{E}_0 &= \mu_0 \varepsilon_0 \frac{\partial^2 \mathrm{E}_0}{\partial t^2}, \ x \leq 0, \\ \nabla^2 \mathrm{E}_1 &= \mu_0 \varepsilon_1 \frac{\partial^2 \mathrm{E}_1}{\partial t^2}, \ x > 0, \\ \nabla^2 \mathrm{B}_0 &= \mu_0 \varepsilon_0 \frac{\partial^2 \mathrm{B}_0}{\partial t^2}, \ x \leq 0, \\ \nabla^2 \mathrm{B}_1 &= \mu_0 \varepsilon_1 \frac{\partial^2 \mathrm{B}_1}{\partial t^2}, \ x > 0, \end{aligned} \quad (20)$$

The boundary condition at $x = 0$ is

$$\begin{aligned} \varepsilon_1 E_{1x}(0^+, y, z, t) &= \varepsilon_0 E_{0x}(0, y, z, t), \\ E_{1y}(0^+, y, z, t)\mathrm{e}_y + E_{1z}(0^+, y, z, t)\mathrm{e}_z &= E_{0y}(0^+, y, z, t)\mathrm{e}_y + E_{0z}(0^+, y, z, t)\mathrm{e}_z, \\ \mathrm{B}_1(0^+, y, z, t) &= \mathrm{B}_0(0, y, z, t). \end{aligned} \tag{21}$$

The general form of solution equation (12) can be written as the superposition of a series of monochromatic plane waves (for simplicity, only the Electric field is discussed below, the derivation for the Magnetic solution is similar)

$$\mathrm{E}(\mathrm{r}, t) = \sum_n \tilde{\mathrm{E}}_n e^{i(\mathrm{k}_n \cdot \mathrm{r} - \omega_n t)}, \tag{22}$$

where $\tilde{\mathrm{E}}_n$ is the constant amplitude, k_i is the wave vector, $\mathrm{r} = x\mathrm{e}_x + y\mathrm{e}_y + z\mathrm{e}_z$ is the position vector, and ω is the frequency which is related to the wave vector by

$$\omega_i = \frac{1}{\sqrt{\mu_0 \varepsilon}} k_i. \tag{23}$$

Without loss of generality, we consider a monochromatic plane wave $\mathcal{W}$ (with $\tilde{\mathrm{E}}_I, \mathrm{k}_I, \omega$) propagating from ε_0 region to ε_1 region. At the boundary $x = 0$, part of $\mathcal{W}\big(\tilde{\mathrm{E}}_R, \mathrm{k}_R, \omega\big)$ is reflected, while the other part transmits $\big(\tilde{\mathrm{E}}_T, \mathrm{k}_T, \omega\big)$, which means the solution to equation (20) is

$$\begin{aligned} \mathrm{E}_0 &= \tilde{\mathrm{E}}_I e^{i(\mathrm{k}_I \cdot \mathrm{r} - \omega t)} + \tilde{\mathrm{E}}_R e^{i(\mathrm{k}_R \cdot \mathrm{r} - \omega t)}, \\ \mathrm{E}_1 &= \tilde{\mathrm{E}}_T e^{i(\mathrm{k}_T \cdot \mathrm{r} - \omega t)}. \end{aligned} \tag{24}$$

The frequency fulfils

$$\omega = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} k_I = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} k_R = \frac{1}{\sqrt{\mu_0 \varepsilon_1}} k_T. \tag{25}$$

The boundary condition equation (21) requires

$$\mathrm{k}_I \cdot \mathrm{r}|_{x=0} = \mathrm{k}_R \cdot \mathrm{r}|_{x=0} = \mathrm{k}_T \cdot \mathrm{r}|_{x=0}. \tag{26}$$

This means (with setting the coordinates such that k_I is in the xz plane)

$$\begin{aligned} \mathrm{k}_I \cdot \mathrm{e}_z &= \mathrm{k}_R \cdot \mathrm{e}_z = \mathrm{k}_T \cdot \mathrm{e}_z, \\ \mathrm{k}_R \cdot \mathrm{e}_y &= \mathrm{k}_T \cdot \mathrm{e}_y = 0, \end{aligned} \tag{27}$$

so k_R and k_T are also in the xz plane. By defining the angle between $\mathrm{k}_I, \mathrm{k}_R, \mathrm{k}_T$ and x axis as $\theta_I, \theta_R, \theta_T$, respectively, we reformulate the boundary condition as

$$\begin{aligned} k_I = k_R &= \frac{1}{\sqrt{\epsilon_1/\epsilon_0}} k_T, \\ k_I \sin\theta_I = k_R \sin\theta_R &= k_T \sin\theta_T, \\ \varepsilon_0\big(-\tilde{E}_I \sin\theta_I + \tilde{E}_R \sin\theta_R\big) &= -\varepsilon_1 \tilde{E}_T \sin\theta_T, \\ \tilde{E}_I \cos\theta_I + \tilde{E}_R \cos\theta_R &= \tilde{E}_T \cos\theta_T, \end{aligned} \tag{28}$$

from which we obtain the reflection and refraction laws of light.

4. Discussion

The entire solution discussed above relies on the prerequisite condition that the function $U(x, y, z, t)$ can be written in the form of $X(x)Y(y)Z(z)T(t)$, which was previously assumed to be true in this paper. This particular claim is regarded as justifiable for two reasons. One, the supposed form, when substituted back into equation (1), results in a meaningful, valid solution. Two, it can be verified strictly that the function $U(x, y, z, t)$ does meet the Dirichle conditions and thus can be expanded as a Fourier series, which arises precisely in the assumed form. Hence, the method of separation of variables itself is justified.

5. Conclusion

In conclusion, by separating variables where appropriate, we have obtained the solution to the generalised wave equation and applied it to the analysis of electromagnetic waves in linear Dielectric media. Specific rules that govern the behaviour of light (e.g., the laws of reflection and refraction), which are merely perceived by many as patterns or properties from observation, are also established strictly from mathematical derivation. While the method of separation of variables proves to be effective in some circumstances, it shall be made abundantly clear that to fully understand the implications of waves in specific mediums with more complex and less ideal conditions, in reality, novel approaches that provide access to reduced complexity could be investigated.

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